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PROBLEMS IN INFINITE SERIES AND DEFINITE INTEGRALS; WITH A STATEMENT OF CERTAIN SUFFICIENT CONDITIONS WHICH ARE FUNDAMENTAL IN THE THEORY OF DEFINITE INTEGRALS.

BY W. F. OSGOOD.

SIMPLE sufficient conditions that a convergent series of continuous real functions of a real variable (or variables)

$$u_1(x) + u_2(x) + \cdots$$

may represent a continuous function and may be integrated or differentiated term by term are now pretty generally known;—I refer to the conditions stated at the beginning of §1. That equally simple conditions, closely analogous to these, exist for the *Definite Integral*

$$\int_a^\infty f(x, \alpha)\, dx$$

is less generally known and it is believed that a statement of these conditions will be welcome to the readers of the ANNALS.* Here α shall lie in the interval (a, b) and $f(x, \alpha)$ shall be a continuous function of the two independent real variables x, α for all values of these variables belonging to the domain

$$S: \quad \begin{cases} a \leqq \alpha \leqq b, \\ g \leqq x. \end{cases}$$

It is assumed that the integral converges for every value of α in the interval (a, b); *i. e.* that, α being given any such value and then held fast, the integral

$$\int_g^x f(x, \alpha)\, dx$$

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* These conditions were obtained by Ch.-J. de la Vallée-Poussin in a noted paper: Etude des intégrales à limites infinies pour lesquelles la fonction sous le signe est continue, *Annales de la Société scientifique de Bruxelles*, vol. 16 (1891/92), p. 150; continued in *Journal de mathématiques*, ser. 4, vol. 8 (1892), p. 421. *Cf.* also C. Jordan, *Cours d'analyse*, vol. 2, 2d ed., §§64–72; O. Stolz, *Diff.- u. Integralrechnung*, vol. 3, chap. 16.

It is to be regretted that the late Professor Brunel in his excellent article on Definite Integrals, *Encyklopädie der mathematischen Wissenschaften*, vol. 2, p. 135, did not state these conditions. He gives but one of them, and that, one of the more elaborate for dealing with the special integral of §3 below.

will approach a limit when x becomes infinite. This limit will be a function of a; we write it as $\phi(a)$. By definition, then,

$$\phi(a) = \lim_{x=\infty} \int_g^x f(x, a)\,dx = \int_g^\infty f(x, a)\,dx.$$

The questions which present themselves are:

(a) When will the function $\phi(a)$ be continuous?

(b) If continuous, when can it be integrated by integrating the integral under the sign of integration, *i. e.* when will

$$\int_{a_0}^{a} \phi(a)\,da = \int_g^{x} dx \int_{a_0}^{a} f(x, a)\,dx \quad ?$$

(c) When can $\phi(a)$ be differentiated by differentiating the integral under the sign of integration; *i. e.* when will

$$\frac{d\phi(a)}{da} = \int_g^\infty \frac{\partial f(x, a)}{\partial a}\,dx \quad ?$$

In the great majority of cases that arise in practice the answer to these questions is not difficult. It is stated below in §2.

§1.

The Tests for an Infinite Series.

Theorem A. *If each term of the series*

$$u_1(x) + u_2(x) + \cdots$$

is a continuous function of the real variable x in the interval $a \leqq x \leqq b$ and if the series converges uniformly in this interval, then the function $f(x)$ represented by the series is continuous in this interval.

The Weierstrassian M-Test. *The above series will converge uniformly if there exists a convergent series of positive constants, $\sum\limits_{n=1}^{\infty} M_n$, such that, no matter what value x may have in the interval,*

$$| u_n(x) | \leqq M_n, \qquad n = 1, 2, \cdots$$

Theorem B. *If the above series converges uniformly, it can be integrated term by term:*

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1} u_1(x)\,dx + \int_{x_0}^{x_1} u_2(x)\,dx + \cdots,$$

where x_0, x_1, are any two numbers of the interval (a, b).

THEOREM C. *If each term of the convergent series*

$$f(x) = u_1(x) + u_2(x) + \cdots$$

has a derivative $u'_n(x)$ *which is continuous throughout the interval* $a \leqq x \leqq b$ *and if the series of the derivatives*

$$u'_1(x) + u'_2(x) + \cdots$$

converges uniformly throughout this interval, then $f(x)$ *has a derivative at each point of the interval and the derivative is given by the formula:*

$$f'(x) = u'_1(x) + u'_2(x) + \cdots .$$

Each of the above four conditions is sufficient; no one of them is necessary. Theorem B is not in general true if a or b is infinite and the integral is extended to infinity.

§2.

THE TESTS FOR A DEFINITE INTEGRAL.

Let $f(x, \alpha)$ be a continuous function of the real variables x, α at all points of the region

$$S: \quad \begin{cases} a \leqq \alpha \leqq b, \\ g \leqq x. \end{cases}$$

The integral

$$\int_g^h f(x, \alpha)\, dx,$$

h being a constant, is a *proper integral* and it defines a continuous function of α, $\phi(\alpha)$. This appears at once if we interpret the integral geometrically as the area under the curve in which the surface $z = f(x, y)$ is cut by the plane $y = \alpha$.*

If, however, the upper limit of integration is infinite, we have to do with an *improper integral*, and it by no means follows, even if this integral:

$$\int_g^\infty f(x, \alpha)\, dx$$

converges for all values of α in question, that the function which it represents, $\phi(\alpha)$, is continuous. For example, we see by direct computation that the integral

$$\int_0^\infty \alpha e^{-\alpha x}\, dx$$

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* For an analytic proof *cf.* Picard, *Traité d'analyse*, vol. 1, p. 29.

converges for all values of $\alpha \geqq 0$; but the function which it represents, $\phi(x)$, is discontinuous:

$$\begin{aligned} \phi(\alpha) &= 1, \qquad & \alpha > 0, \\ \phi(\alpha) &= 0, \qquad & \alpha = 0. \end{aligned}$$

Again, the integral

$$\int_0^\infty \sin \pi\alpha \, e^{-\sin^2 \pi\alpha x} \, dx,$$

obtained from the foregoing in a simple manner, converges for all values of α; but it is discontinuous for every positive or negative integral value of α, inclusive of zero.

Under what conditions, then, will the integral

$$\int_g^\infty f(x, \alpha) dx,$$

assumed convergent, represent a continuous function of α? The answer to this and similar questions is given in the theorems that follow.

Definition of Uniform Convergence of an Improper Definite Integral. The integral

$$\int_g^\infty f(x, \alpha) dx,$$

where $f(x, \alpha)$ is continuous throughout the region S, is said to *converge uniformly* throughout the interval $a \leqq \alpha \leqq b$ if, a positive quantity ϵ being chosen at pleasure, there exists a positive quantity G, independent of α, such that

$$\left| \int_G^x f(x, \alpha) dx \right| < \epsilon, \qquad x > G.$$

THEOREM A. *If $f(x, \alpha)$ is a function of the independent real variables x, α continuous throughout the region S, and if the integral*

$$\int_g^\infty f(x, \alpha) dx$$

converges uniformly throughout the interval $a \leqq \alpha \leqq b$, then the function $\phi(\alpha)$ represented by the integral is continuous throughout this interval.

THE VALLÉE-POUSSIN $\mu(x)$-TEST. *The above integral will converge uniformly if there exists a function* $\mu(x)$, *independent of* α, *which satisfies the following conditions:*

$$(a) \qquad \mu(x) \geqq 0, \qquad x \geqq g;$$

$$(b) \qquad |f(x, \alpha)| \leqq \mu(x), \qquad x \geqq g;$$

$$(c) \qquad \int_g^\infty \mu(x)\, dx \quad \textit{converges.}$$

COROLLARY. *These conditions will always be satisfied if there exists a constant* $k > 1$ *such that the product*

$$x^k f(x, \alpha)$$

remains finite throughout the region S.

THEOREM B. *If the above integral converges uniformly, it can be integrated under the sign of integration:*

$$\int_{\alpha_0}^{\alpha_1} \phi(\alpha)\, d\alpha = \int_g^\infty dx \int_{\alpha_0}^{\alpha_1} f(x, \alpha)\, d\alpha,$$

where α_0, α_1, *are any two numbers of the interval* (a, b).

THEOREM C. *If the above function has a first partial derivative with regard to* α, $f_\alpha(x, \alpha)$, *at each point of* S, *continuous throughout* S, *and if*

$$\int_g^\infty f_\alpha(x, \alpha)\, dx$$

converges uniformly throughout the interval $a \leqq \alpha \leqq b$; *then the function* $\phi(\alpha)$ *has a derivative at each point of the interval, and*

$$\frac{d\phi(\alpha)}{d\alpha} = \int_g^\infty f_\alpha(x, \alpha)\, dx.$$

Each of the above four conditions is sufficient; no one of them is necessary. Theorem B is not in general true if a or b is infinite and the integral with regard to α is extended to infinity.

Other Improper Integrals. In the foregoing, improper integrals were considered, one of whose limits of integration is infinite, but whose integrand is continuous throughout S. Similar theorems hold for integrals taken between finite limits, the integrand becoming infinite at one of the limits of integration.

Thus, let $f(x, \alpha)$ be a continuous function of x, α at all points of the region

$$R : \quad \begin{cases} a \leqq \alpha \leqq b, \\ g \leqq x < h; \end{cases}$$

but suppose that $f(x, \alpha)$ becomes infinite in the points $x = h$, $\alpha = a$, the integral now being

$$\int_g^h f(x, \alpha) dx.$$

Then the definition of uniform convergence will read, *mutatis mutandis*, precisely as before except that G is now taken near to h and x lies between G and h. The four tests hold as before, with obvious modifications.

The proofs of the foregoing theorems are similar to the proofs of the corresponding theorems of §1 and serve as a useful exercise for the student who is familiar with these proofs and is just beginning the study of definite integrals from a modern standpoint.

§3.

A SPECIAL SERIES AND A SPECIAL INTEGRAL.

I. *The Series.* Let each term of the series

$$u_1(x) + u_2(x) + \cdot \cdot \cdot$$

be continuous for all values of $x \geqq g$ and let the series converge uniformly throughout any arbitrarily assigned interval $g \leqq x \leqq h$. Then, if $f(x)$ denotes the function represented by the series, $f(x)$ is continuous and

$$\int_g^x f(x) dx = \int_g^x u_1(x) dx + \int_g^x u_2(x) dx + \cdot \cdot \cdot \cdot .$$

When can this series be integrated term by term to infinity? We must, of course, first of all require that each term be integrable to infinity; *i. e.*

1° $\int_g^\infty u_n(x) dx$ shall converge, $\qquad n = 1, 2, \ldots$

Furthermore let the above series of integrals converge uniformly throughout the *unlimited* interval $x \geqq g$; *i. e.*

2° A positive quantity ϵ being chosen arbitrarily small, it shall then be possible to find a positive integer independent of x such that for all values of $x \geqq g$.

$$\left| \int_g^x u_{m+1}(x)dx + \cdot \cdot \cdot + \int_g^x u_{m+p}(x)dx \right| < \epsilon, \qquad p = 1, 2 \cdot \cdot \cdot$$

Then it is easy to show that the series can be integrated to infinity term by term; or, more fully,

(α) that the series of limits

$$\int_g^\infty u_1(x)dx + \int_g^\infty u_2(x)dx + \cdot \cdot \cdot$$

converges;

(β) that the integral $\int_g^\infty f(x)\, dx$ converges;

(γ) that

$$\int_g^\infty f(x)dx = \int_g^\infty u_1(x)dx + \int_g^\infty u_2(x)dx + \cdot \cdot \cdot$$

II. *The Integral.* A closely analogous sufficient condition holds for the reversal of the order of integration in the integral

$$\int_a^\infty d\alpha \int_g^\infty f(x, \alpha)dx,$$

where $f(x, \alpha)$ is a continuous function at all points (x, α) of the region

$$\bar{S} : \quad a \leqq \alpha, \quad g \leqq x.$$

1° *The integrals*

$$\int_g^\infty f(x; \alpha)dx, \qquad \int_a^\infty f(x, \alpha)d\alpha$$

shall respectively converge uniformly throughout every FIXED *interval*;

$$a \leqq \alpha \leqq b, \qquad g \leqq x \leqq h;$$

2° *The integral*

$$\int_a^\alpha d\alpha \int_g^\infty f(x, \alpha)dx$$

shall converge uniformly throughout the UNLIMITED *interval* $\alpha \geqq a$; *i. e. a positive quantity* ϵ *being chosen arbitrarily small, it shall then be possible to find a positive quantity* G, *independent of* α, *such that for all values of* $\alpha \geqq a$

$$\left| \int_a^\alpha d\alpha \int_G^x f(x, \alpha) dx \right| < \epsilon, \qquad x > G.$$

When these conditions are fulfilled

$$\int_a^\infty d\alpha \int_g^\infty f(x, \alpha) dx = \int_g^\infty dx \int_a^\infty f(x, \alpha) d\alpha.$$

It sometimes happens that one of the integrals of Condition 1°,—the first, let us say,—does not converge uniformly in the interval (a, b), but that it converges uniformly in every interval lying inside of the region $\alpha \geqq a$, namely, $a < a' \leqq \alpha \leqq b$, where a', b are arbitrary.

In this case, if the remainder of Condition 1° and Condition 2° are satisfied, the reversal of the order of integration is still allowable.

A more elaborate test for a case of importance in practice is also given by Vallée-Poussin and is stated in the *Encyklopädie, l. c.*

§4.

PROBLEMS IN INFINITE SERIES AND PRODUCTS.

A.—CONVERGENCE.

1. Is the infinite series:

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 + \cdot \cdot \cdot$$

convergent or divergent?

2. Obtain by means of Cauchy's integral test for the convergence of an infinite series of positive terms (*cf.* Picard, *Traité d'analyse*, vol. 1, p. 27) an upper limit for the value of the remainder of the series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdot \cdot \cdot, \qquad p > 1.$$

Hence compute the value of this series when $p = 1\frac{1}{2}$ correct to two places of decimals. How many terms of this series would it be necessary to sum in order to obtain the same degree of approximation?

3. For what values of x does the series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt[n]{n+n^n}}$$

converge? Discuss all cases.

4. Determine all values of x for which the series

$$\left(\frac{1}{3}\right)^2 x - \left(\frac{1\cdot 2}{3\cdot 5}\right)^2 x^2 + \left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2 x^3 \cdots$$

converges.

5. Is the series

$$\sum_{n=1}^{\infty} \frac{1}{\log(1+n)\cdot\log(1+n^n)}$$

convergent or divergent?

6. Show that if

$$a_0 + a_1x + a_2x^2 + \cdots$$

represents a power series convergent for values of x other than $x = 0$, the series

$$a_0 + a_1x + \frac{1}{2!}a_2x^2 + \quad \cdots + \frac{1}{i!}a_ix^i + \cdots$$

will converge for all values of x.

For what values of x will the series

$$\sum_{i=1}^{\infty} \frac{1}{i^2} a_ix^i$$

converge?

B.—Series of Functions.

7. Let the series

$$u_1(x) + u_2(x) + \cdots\cdot$$

be a series of functions each continuous in the interval $a \leqq x \leqq b$ and let it converge uniformly in this interval. Show that the term-by-term integral of this series:

$$\int_a^x u_1(x)\,dx + \int_a^x u_2(x)\,dx + \cdots\cdot$$

is a uniformly convergent series throughout this interval.

8. Let $s_n(m)$ be a single valued function of the two positive integers m, n, which satisfies the following conditions:

(*a*) $\lim\limits_{n=\infty} s_n(m)$ exists; denote it by $f(m)$;

(*b*) $\lim\limits_{m=\infty} s_n(m)$ exists; denote it by S_n;

(*c*) $s_n(m)$ converges uniformly when $n=\infty$:

$$|\, s_n(m) - s_{n'}(m)\, | < \epsilon, \qquad n > \nu,\ n' > \nu.$$

Then

(1) $\lim\limits_{m=\infty} f(m)$ exists; denote it by A;

(2) $\lim\limits_{n=\infty} S_n$ exists; denote it by B;

(3) $A = B$.

9. Apply the theorem of Problem 8 to prove that

$$\lim_{n=\infty}\left(1+\frac{x}{n}\right)^n = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \cdots$$

10. From the formula

$$\cos mx = \cos^m x - \frac{m(m-1)}{1\cdot 2}\cos^{m-2}x\sin^2 x + \cdots\cdot$$

deduce the development

$$\cos x = 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \cdots\cdot$$

and justify each step of your work.

11. The series

$$\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \cdots$$

converges when $x > 1$. Does the series represent a continuous function? Can it be differentiated term by term? Denoting the function represented by the series by $f(x)$, discuss rigorously the shape of the curve $y = f(x)$.

12. Let

$$a_0 + a_1 + \cdots$$

be an absolutely convergent series of constant terms, and let

$$f_0(x),\ f_1(x),\ \cdots$$

be a set of functions each continuous within the interval $\alpha \leqq x \leqq \beta$ and each comprised between certain fixed limits

$$A \leqq f_i(x) \leqq B, \qquad i = 0, 1, \ldots$$

where A, B are constants. Show that the series

$$a_0 f_0(x) + a_1 f_1(x) + \cdots$$

represents a continuous function of x within this interval: $\alpha \leqq x \leqq \beta$.

13. Let $s_n(x)$ be a continuous function of x in the interval $a \leqq x \leqq b$ and let $s_n(x)$ converge, for every rational value of x pertaining to this interval, toward a limit when n increases indefinitely. Furthermore, let

$$| s_n(x') - s_n(x) | < M | x' - x | ,$$

where M is a fixed positive quantity and x, x' are any two points of the interval. Prove that $s_n(x)$ converges uniformly toward a limit for all values of x in the interval.

14. If $$a_0 + a_1 + \cdots$$
and $$b_0 + b_1 + \cdots$$
are any two absolutely convergent series, show that the value of the series

$$a_0 b_0 + a_1 b_1 + \cdots$$

is given by the integral:

$$\frac{1}{2\pi} \int_0^{2\pi} f_1(e^{\theta i})\, f_2(e^{-\theta i})\, d\theta,$$

where

$$f_1(x) = \sum_0^{\infty} a_i x^i, \qquad f_2(x) = \sum_0^{\infty} b_i x^i.$$

15. If the terms of the series

$$u_1(x) + u_2(x) + \cdots$$

are all uniformly continuous throughout the interval $a < x < b$ and if the series converges uniformly throughout this interval, show that the function $f(x)$ represented by the series approaches a limit when x approaches a.

16. Given that the series

$$u_1(x) + u_2(x) + \cdots$$

converges uniformly throughout the interval $a < x < b$ and that it converges when $x = a$. Can we infer that it converges uniformly when $a \leqq x < b$?

17. If the terms of the series

$$u_1(x) + u_2(x) + \cdots$$

are all continuous throughout the interval $a \leqq x \leqq b$ and if the series converges uniformly throughout the interval $a < x \leqq b$, will it necessarily converge when $x = a$, and, if it does converge, will the function that it represents necessarily be continuous at the point $x = a$?

18. If the terms of the series

$$u_1(x) + u_2(x) + \cdots$$

are all continuous throughout the interval $a \leqq x \leqq b$ and if the series converges uniformly throughout every interval $\alpha \leqq x \leqq \beta$ lying inside of the interval (a, b) : $a < \alpha < \beta < b$, will the series necessarily converge when $x = a$, and, if it does converge, will the function that it represents necessarily be continuous at the point $x = a$?

19. Show that the series

$$\frac{1}{\sqrt{1 - 2r\mu + r^2}} = \sum_0^\infty r^n P_n(\mu), \qquad \left\{ \begin{array}{l} -1 \leqq \mu \leqq 1, \\ -1 < r < 1, \end{array} \right.$$

can be differentiated term by term both as regards r and as regards μ.

Suggestion: Prove the theorem first for the value of μ in question and for a value of r numerically small; and then show that the theorem still holds when r has the value in question.

20. Given the formula

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left\{ 1 - \frac{x^2}{2 \cdot (2n+2)} + \frac{x^4}{2 \cdot 4 \cdot (2n+2)(2n+4)} - \cdots \right\}$$

show that

$$\frac{\partial^2 J_n(x)}{\partial n \, \partial x} = \frac{\partial^2 J_n(x)}{\partial x \, \partial n}.$$

21. Show that if the series

$$V = \phi_0(x, y, z) + \phi_1(x, y, z) + \phi_2(x, y, z) + \cdots,$$

where $\phi_n(x, y, z)$ denotes a homogeneous polynomial in (x, y, z) of degree n, represents a function satisfying Laplace's equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

then each function $\phi_n(x, y, z)$ must also satisfy Laplace's equation.

Suggestion: Assume that such a function V can be developed into a triple power series in x, y, z for values of the arguments numerically sufficiently restricted, the development holding for all values of the arguments in the neighborhood of the point $(0, 0, 0)$.

22. Show that if

$$u_0(x) + u_1(x) + \cdots$$

is a series of functions each continuous and having no roots in the interval $a \leqq x \leqq b$, and if

$$\left| \frac{u_{n+1}(x)}{u_n(x)} \right| \leqq \gamma < 1, \qquad n \geqq m,$$

where γ, m, do not depend on x, then the given series is uniformly convergent within the interval.

Apply this test to the following series:

$$1 + xa + \frac{x(x-1)}{2!}a^2 + \frac{x(x-1)(x-2)}{3!}a^3 + \cdots$$

where $0 < a < 1.$

23. Find all the values of x for which the series

$$e^x \sin x + e^{2x} \sin 2x + e^{3x} \sin 3x + \cdots$$

converges. Does it converge uniformly for these values?

For what values of x can the series be differentiated term by term?

24. For what values of x is the series

$$1 + x + x^2 + \cdots$$

absolutely convergent? Is it uniformly convergent throughout the same interval?

25. Does the series

$$x + \frac{x^3}{3} - \frac{x^2}{2} + \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^4}{4} + \frac{x^9}{9} + \frac{x^{11}}{11} - \frac{x^6}{6} + \cdots$$

converge uniformly in the interval $0 \leqq x \leqq 1$?

26. Show that the series

$$\frac{1}{(1+x)^2} + \frac{1}{(2+x)^2} + \frac{1}{(3+x)^2} + \cdots, \qquad 0 \leqq x,$$

can be integrated term by term between any two positive finite limits.

Can the series be integrated term by term between the limits 0 and ∞ ? Can the function defined by the series be integrated between these limits?

27. Show that the series

$$2xe^{-x^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2} e^{-\frac{x^2}{n^2}} - \frac{1}{(n+1)^2} e^{-\frac{x^2}{(n+1)^2}} \right)$$

can be integrated term by term between any two finite limits.

Can the function defined by the series be integrated between the limits 0 and ∞ ? If so, is the value of this integral given by integrating the series term by term between these limits?

28. If each term of the series

$$u_1(x) + u_2(x) + \cdots$$

is a continuous function of x for all values of $x \geqq a$, and if the series

$$x^k u_1(x) + x^k u_2(x) + \cdots, \qquad k > 1,$$

satisfies the M-test (§ 1), then the given series may be integrated term by term from a to ∞ .

29. Show that the series

$$e^{-x^2} - \frac{x^2 e^{-x^2}}{2\,!} + \frac{x^4 e^{-x^2}}{4\,!} - \frac{x^6 e^{-x^2}}{6\,!} + \cdots$$

can be integrated term by term from 0 to ∞, given that $(n-1)\,! = n^{n-\frac{1}{2}} e^{-n} K_n$, when K_n lies between two positive constant quantities.

30. Can the function defined by the series

$$\frac{1}{(1+x)^3} + \frac{1}{(2+x)^3} + \frac{1}{(3+x)^3} + \cdot$$

be integrated from 0 to ∞ , and if so, is its value given by the term-by-term integral of the series?

C.—DOUBLE SERIES.

31. Show that the function

$$\log(1 + e^x)$$

can be developed into a power series in x. State carefully each theorem that you employ.* Determine the coefficients of the first four terms in the development.

32. Develop the function

$$f(x) = \int_0^{\frac{\pi}{2}} \sin^{-1}(x \sin\phi)\, d\phi$$

into a power series in x. Justify each step. Assume the formula

$$\int_0^{\frac{\pi}{2}} \sin^n \phi \, d\phi = \frac{2 \cdot 4 \cdot 6 \cdot \, \ldots \, \cdot (n-1)}{1 \cdot 3 \cdot 5 \cdot \, \ldots \, \cdot n},$$

where n is odd.

33. Develop $\sqrt[3]{\cos x}$ into a series proceeding according to ascending integral powers of x, stating accurately the theorems of which you make use. For how large a region will the series surely converge and represent the function?

Further problems in infinite series will be found in the exercises at the end of chs. 17 and 18 of Gibson's *Elementary Treatise on the Calculus*.

D.—INFINITE PRODUCTS.

34. Discuss the infinite product

$$(1 - \sin x)\,(1 + \sin \tfrac{1}{2}x)\,(1 - \sin \tfrac{1}{3}x) \;\ldots\ldots$$

with reference to absolute and to uniform convergence.† Study the function defined by this product and plot the corresponding curve.

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* The final theorem here needed is to be found in Stolz, *Allgemeine Arithmetik*, vol. 1, ch. 10, §25. A special case of the theorem sufficiently general for the great majority of the cases that arise in practice (in particular, for the present case) is stated in my *Infinite Series*, §36.

Numerous examples of developments that can be obtained by this theorem are given in the text-books on Calculus; *cf.*, for example, Byerly, *Problems in Differential Calculus*, pp. 46–48. The analysis at the disposal of the student at that stage of his work is inadequate for complete demonstrations, and these are properly deferred for an advanced course in analysis.

† An infinite product $f_1 f_2 \ldots\ldots$ is said to converge when and only when (*a*) the factors from a definite one, f_m, on are all different from zero; (*b*) the product $f_{m+1} f_{m+2} \cdots f_{m+p}$ approaches a limit different from zero.

An infinite product $f_1(x)\, f_2(x) \ldots$ is said to converge uniformly throughout the interval $a \leqq x \leqq b$ if, a positive quantity ϵ being chosen at pleasure, there exists a positive integer m independent of x and such that

$$| f_{m+1}(x)\, f_{m+2}(x) \;\ldots\; f_{m+p}(x) - 1 \,| < \epsilon, \qquad p = 1, 2, \ldots\ldots$$

These definitions can, of course, be replaced by more general ones. But the products that would thus be admitted it is not useful to consider in practice.

35. Show that the infinite product

$$\cos x \cdot \cos \tfrac{1}{2} x \cdot \cos \tfrac{1}{3} x \cdot \ . \ . \ .$$

converges for all values of x and discuss the function that it defines.

36. Discuss the infinite product

$$\left(1 - \frac{x}{1}\right)\left(1 + \frac{x}{3}\right)\left(1 - \frac{x}{5}\right)\left(1 + \frac{x}{7}\right) \cdot \cdot \cdot \cdot$$

with regard to absolute and to uniform convergence. Plot the function defined by the product as a curve. Find its slope when $x = 0$ and when $x = 1$.

37. Show that the above products can be developed into power series in x and determine the coefficients of the terms of degree 0, 1, 2.

§5.

Problems in Definite Integrals.

38. Show that the integral

$$\int_1^\infty x^{a-1} e^{-x}\, dx$$

converges and defines a continuous function for all values of a.

39. Show that the integral

$$\int_0^1 x^{a-1} e^{-x}\, dx$$

converges and defines a continuous function for all positive values of a.

Give the proof first by introducing as a new variable of integration $y = x^{-1}$; then by applying the theorems of §2 directly, modified for a finite domain of integration.

40. Show that the function

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x}\, dx$$

is continuous for all positive values of a and that

$$\Gamma'(a) = \int_0^\infty x^{a-1} \log x e^{-x}\, dx$$

$$\Gamma''(a) = \int_0^\infty x^{a-1} (\log x)^2 e^{-x}\, dx$$

etc.

Plot the curve $y = \Gamma(a)$, showing that it has a minimum between $a = 1$ and $a = 2$, and that it is always convex toward the a-axis.

41. The integral

$$\int_0^\infty a^3 e^{-a^2 x} dx$$

converges for all values of a and represents the function $\phi(a) = a$. Can the integral be differentiated under the sign of integration?

42. Given the formula

$$\int_0^\infty e^{-x^2} dx = \tfrac{1}{2}\sqrt{\pi},$$

deduce the formulas

$$\int_0^\infty e^{-ax^2} dx = \tfrac{1}{2}\sqrt{\frac{\pi}{a}},$$

$$\int_0^\infty x^2 e^{-ax^2} dx = \tfrac{1}{4}\frac{\sqrt{\pi}}{a^{\frac{3}{2}}}, \text{ etc.}$$

43. Show that

$$\int_0^\infty e^{-x^2} \cos 2ax \, dx = \tfrac{1}{2}\sqrt{\pi}\, e^{-x^2}$$

Suggestion: Denote the value of the function by u and show that

$$\frac{du}{da} + 2au = 0, \qquad u|_{a=0} = \tfrac{1}{2}\sqrt{\pi}.$$

44. Show that

$$\int_0^\infty e^{-\left(x^2 + \frac{a^2}{x^2}\right)} dx = \tfrac{1}{2}\pi e^{-2a}.$$

Is it allowable here to differentiate under the sign of integration for all values of a? If not, and if you bave used the method suggested in Problem 43, have you been careful to make your solution rigorous?

45. Show 1) that the integral

$$\int_0^\infty \frac{\cos ax}{1 + x^2} dx$$

defines a continuous function, $f(a)$, for all values of a; 2) that

$$\phi(a) = \int_0^a f(a)\, da$$

is given by integration under the integral sign with respect to a; 3) that $f'(a)$ is given in general by differentiating under the integral sign with respect to a; 4) that $\phi(a)$ satisfies the differential equation:

$$\frac{d^2\phi}{da^2} = \phi - \frac{\pi}{2} \qquad \text{when } a > 0.$$

Hence determine the function $f(a)$ defined by the integral.

46. Discuss the function defined by the integral:

$$\int_0^\infty e^{-a^2x^2} \sin x \, dx.$$

47. Discuss the function

$$\phi(a) = \int_0^{\frac{\pi}{2}} \tan^a \phi \, d\phi,$$

plotting the graph and determining the essential characteristics.

48. Show that the functions

$$f(a) = \int_0^\infty e^{-x^2} \cos(ax^2) dx,$$

$$\phi(a) = \int_0^\infty e^{-x^2} \sin(ax^2) dx,$$

satisfy the linear differential equations:

$$f'(a) = \frac{-1}{2(1+a^2)} \Big[a f(a) + \phi(a) \Big],$$

$$\phi'(a) = \frac{1}{2(1+a^2)} \Big[f(a) - a\phi(a) \Big].$$

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